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Summary Of Maximum Theoretical Accuracy Of Radar Measurements

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Abstract: This paper summarizes some general formulas for maximum theoretical accuracy of radar measurements on a target in the presence of additive white Gaussian noise. The formulas are specialized to some particular cases of interest.

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SUMMARY OF MAXIMUM THEORETICAL ACCURACY OF RADAR MEASUREMENTS

INTRODUCTION

The parameters of a radar target determine how the transmitted waveform is modified by the target to produce a reflected waveform. During a short time interval the target is completely characterized by several parameters: cross section, range, radial velocity, angular position, and (if necessary) angular rate and radial acceleration. Once the target has been detected, the signal energy received across the antenna aperture may be processed to form an estimate of the target parameters. We assume that the received waveform has been corrupted by additive white Gaussian noise. An optimum estimation method for measuring the target parameters is the *method of maximum likelihood*. This method can be implemented in the case of additive white Gaussian noise by selecting the peak output from a set of filters where each filter is matched to a set of parameter values. There is one filter for each essentially different combination of parameter values. The accuracy available from the maximum likelihood method depends not only on the relative strength of signal and noise at the receiver, but also on the shape of the antenna aperture and the character of the transmitted waveform. This report contains the accuracy formulas which have been derived. Free space propagation is assumed throughout.

The transmitted waveform, denoted $x(t)$, is related to its Fourier transform $X(f)$ by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi i f t}\, df, \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t}\, dt \tag{1}$$

We make the following definitions:

$$\begin{aligned} E &\equiv \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df \\ t_0 &\equiv \frac{1}{E} \int_{-\infty}^{\infty} t\, x^2(t)\, dt \\ f_0 &\equiv \frac{2}{E} \int_{0}^{\infty} f\, |X(f)|^2\, df \end{aligned} \tag{2}$$

where E is called the signal energy (assumed finite), t_0 is the mean time of arrival of the signal, and f_0 is the mean frequency of the signal. Then we can define:

$$\begin{aligned} \beta &\equiv 2\pi \left[\frac{2}{E} \int_{0}^{\infty} (f - f_0)^2\, |X(f)|^2\, df \right]^{1/2} \\ \Delta &\equiv 2\pi \left[\frac{1}{E} \int_{-\infty}^{\infty} (t - t_0)^2\, x^2(t)\, dt \right]^{1/2} \end{aligned} \tag{3}$$

where β and Δ are called the effective bandwidth and the effective time duration of the signal, respectively. N_0 is defined as the noise power per cycle at the receiver, i.e., N_0 is the average noise power at the output of a filter with a one-cycle-wide passband. The quantity $\Re$ is defined by

$$\Re = \frac{2E}{N_0} \tag{4}$$

$\Re$ characterizes the detectability of a known signal in the presence of white Gaussian noise and is exactly the peak signal-to-noise ratio available at the output of a matched filter (reference 1). For good signal detectability, $\Re$ must be fairly large (of the order of 50). The accuracy formulas summarized here depend on the assumption that $\Re$ is large and that ambiguities are not present.

RANGE ACCURACY

Assume that range is the only unknown target parameter. Provided that the envelope of the autocorrelation function of the transmitted waveform can be approximated by a parabola near the origin, the error in the estimation of the time of arrival of the reflected signal is normally distributed with standard deviation δ_t given by (references 2 and 3)

$$\delta_t = \frac{1}{\beta\sqrt{\Re}} \tag{5}$$

The corresponding rms error δ_r in the estimation of range is

$$\delta_r = \frac{c}{2}\,\delta_t = \frac{c}{2\beta\sqrt{\Re}} \tag{6}$$

where c is the velocity of light.

When the transmitted signal has a rectangular envelope, β is infinite and the above formula is not valid. It appears that an exact range accuracy formula for this situation is not simple to obtain. Using an approximate analysis, however, the following formula has been derived for a rectangular pulse of sinusoid with duration T (reference 4).

$$\delta_t = \frac{\sqrt{2}T}{\Re} \quad \text{or} \quad \delta_r = \frac{c}{2}\,\delta_t = \frac{cT}{\sqrt{2}\Re} \tag{7}$$

Rough numerical confirmation of this formula has been obtained. Since $\Re$ is proportional to the product of the peak signal power and T, we have the intuitive obvious result that δ_r for a rectangular pulse depends only on the peak power of the pulse and not on the pulse length.

VELOCITY ACCURACY

Assume that radial velocity is the only unknown parameter of the target. Provided that the envelope of the autocorrelation function of the transmitted waveform, considered as a function of doppler shift, can be approximated by a parabola near the origin (a condition which is usually satisfied), the error in estimation of frequency is normally distributed with standard deviation δ_f given by (reference 4)

$$\delta_f = \frac{1}{\Lambda\sqrt{\mathfrak{R}}} \tag{8}$$

The corresponding rms error δ_v in the estimation of radial velocity is

$$\delta_v = \frac{\lambda}{2}\,\delta_f = \frac{\lambda}{2\Lambda\sqrt{\mathfrak{R}}} \tag{9}$$

where the wavelength $\lambda = c/f_0$.

For the particular case of a rectangular pulse of duration T, $\Lambda = \pi T/\sqrt{3}$ and

$$\delta_v = \frac{\sqrt{3}\,\lambda}{2\pi T\sqrt{\mathfrak{R}}} \tag{10}$$

COMBINED RANGE AND VELOCITY ACCURACY

Assume that both range and radial velocity are unknown target parameters. Provided that the autocorrelation function of the transmitted waveform, considered as both a function of time delay and doppler shift, is well behaved in the manner indicated earlier (this rules out rectangular pulses), we can derive the following expressions for δ_t and δ_f (reference 4), 3

$$\begin{aligned} \delta_t^2 &= \frac{\mathfrak{R}\Delta^2}{(\mathfrak{R}\beta^2)(\mathfrak{R}\Delta^2) - (4A_{12}/N_0)^2} \\ \delta_f^2 &= \frac{\mathfrak{R}\beta^2}{(\mathfrak{R}\beta^2)(\mathfrak{R}\Lambda^2) - (4A_{12}/N_0)^2} \end{aligned} \tag{11}$$

where A_{12} is a quantity which is indicative of the correlation between the errors in measurement of time of arrival and the errors in measurement of frequency. The larger A_{12} is, the larger are the errors δ_t and δ_f. A_{12} is large when the radar waveform is linearly frequency modulated (in one direction only). $A_{12} = 0$ for any waveform which has a center of symmetry.

In the situation of most frequent interest where $A_{12} = 0$, the above expressions for δ_t and δ_f reduce to those which were stated earlier (Eqs. 5 and 8). We see, therefore, that the rms error in the estimation of one of the parameters is not influenced by the lack of knowledge about the other parameter. Generally, the presence of additional unknown parameters can do nothing but increase the error in estimation of any particular parameter. When the parameter in question is not "coupled" to any of these unknown parameters, however, there is no increase in this error.

The lack of coupling between the measurement of target parameters can be extremely useful because it allows us to calculate the rms error in the estimation of each of the parameters separately while ignoring the others. Of the following set of target parameters

1. cross section
2. range
3. radial velocity
4. radial acceleration
5. angle of arrival
6. angular rate

it has been necessary in the derivation of the formulas presented in this paper to include only couplings between the pairs (2, 3) and (2, 4).

ANGULAR ACCURACY

We assume that the reflected waveform from a target arrives across an antenna aperture and that the unknown angle of arrival is approximately normal to the plane of the antenna
4 aperture. The aperture need not necessarily be simply connected, e.g., it may consist of two or more disjoint coplanar areas. It is assumed further that the aperture is large compared to a wavelength but that the maximum difference of time delay from the target and any two points in the aperture is small compared to β^{-1} (the reciprocal bandwidth) of the reflected signal. This latter assumption will be satisfied by signals which are sufficiently narrowband and it insures that angular information is not being derived from coarse range information. The assumption also insures that the estimate of angle of arrival will be uncoupled from the estimates of other target parameters.

From the phase and amplitude information available from the aperture in the presence of receiver noise (and perhaps uniformly distributed background noise), we can form a maximum likelihood estimate of the angle of arrival. The angle of arrival can be represented by two numbers, the components of the angle of arrival measured in each of two reference planes perpendicular to each other and to the plane of the antenna aperture. For apertures of arbitrary shape, the errors in the two measurements will generally be coupled together, but this coupling can be eliminated by choosing the reference planes along the principal axes of the aperture. With this step, the estimate of each component of the angle of arrival is reduced to a one-parameter estimation problem. We assume in what follows that this reduction has been made.

Let ξ and η be the Cartesian coordinates of a point in the aperture. These coordinates are chosen to lie along the principal axes of the aperture with its centroid as origin. Denote the aperture area by S. Then the rms length of the aperture in the ξ direction, $\mathfrak{L}_\xi$, is defined

$$\mathfrak{L}_\xi \equiv 2\pi\left[\frac{1}{S}\int\!\!\int \xi^2\, d\xi\, d\eta\right]^{1/2} \tag{12}$$

$\mathfrak{L}_\eta$ is defined similarly. Note the similarity of these definitions to the definitions of β and Δ (Eq. 3). The general expression for the rms error δ_{θ_ξ} in estimating the angle of arrival θ_ξ, measured in the plane parallel to ξ, is given by*

$$\delta_{\theta_\xi} = \frac{\lambda}{\mathfrak{L}_\xi\sqrt{\mathfrak{R}}} \quad \text{(in radians)} \tag{13}$$

An equivalent result holds for the η plane. We shall find it convenient to drop the subscripts ξ and η and write simply

$$\delta_\theta = \frac{\lambda}{\mathfrak{L}\sqrt{\mathfrak{R}}} \tag{14}$$

where it is assumed that $\mathfrak{L}$ will be measured in a direction parallel to the plane in which θ is defined.

It is useful to adapt Eq. 14 to special cases. For a rectangular aperture of length L, $\mathfrak{L} = \pi L/\sqrt{3}$ and

$$\delta_\theta = \frac{\sqrt{3}}{\pi\sqrt{\mathfrak{R}}}\left(\frac{\lambda}{L}\right) \tag{15}$$

For a circular aperture of diameter D, $\mathfrak{L} = \pi D/2$ and

$$\delta_\theta = \frac{2}{\pi\sqrt{\mathfrak{R}}}\left(\frac{\lambda}{D}\right) \tag{16}$$

For an interferometer system in which the aperture consists of two equal coplanar areas separated by a distance Nλ, which is large compared to the linear dimensions of each aperture, we have $\mathfrak{L} = \pi N\lambda$ and

$$\delta_\theta = \frac{1}{\pi N\sqrt{\mathfrak{R}}} \tag{17}$$

The measurement of θ for the last situation, though it may be very accurate, will usually be highly ambiguous.

* The proof of this formula is indicated in reference 5. See also Eq. 30 of reference 6.

FURTHER ESTIMATES OF TARGET PARAMETERS FROM SETS OF RADAR MEASUREMENTS

Consider the following general problem. We are given a function of time $\mathcal{Z}(t)$ which depends on two unknown parameters γ_1 and γ_2.

$$\mathcal{Z}(t) = \gamma_1 + \gamma_2 t \tag{18}$$

Suppose that we have a series of n ($n \gg 1$) equally spaced measurements of $\mathcal{Z}(t)$, each of which is perturbed by an independent, normally distributed noise with variance σ^2. The time origin $t = 0$ is taken at the center of the measurement interval. (This assumption is made throughout this section.) Then, parameter estimation theory shows that the estimates of γ_1 and γ_2 are uncoupled and have normally distributed errors with standard deviations δ_{γ_1} and δ_{γ_2} given by

$$\delta_{\gamma_1} = \frac{\sigma}{\sqrt{n}}, \quad \delta_{\gamma_2} = \frac{2\sqrt{3}\,\sigma}{\sqrt{n}\,T} \tag{19}$$

where T in these formulas denotes the time duration of the measuring interval (reference 7).

The more general problem

6

$$\mathcal{Z}(t) = \gamma_1 + \gamma_2 t + \gamma_3 t^2 \tag{20}$$

is also of interest. For this problem, the theory tells us that the estimate of γ_2 is uncoupled to the estimate of γ_1 or γ_3, but that the estimates of γ_1 and γ_3 are coupled together, and

$$\begin{aligned} \delta_{\gamma_1} &= \frac{3\sigma}{2\sqrt{n}} \\ \delta_{\gamma_2} &= \frac{2\sqrt{3}\sigma}{\sqrt{n}T} \\ \delta_{\gamma_3} &= \frac{6\sqrt{5}\sigma}{\sqrt{n}T^2} \end{aligned} \tag{21}$$

where as before T denotes the time duration of the measuring interval (reference 7).

We are now ready to apply these results to the problem of combining sets of independent, equally spaced radar measurements.

Suppose the range of a target over a short interval of time can be written

$$r(t) = r_0 + vt + \tfrac{1}{2} a t^2 \tag{22}$$

where the parameters r_0, v, and a are unknown. Suppose also we have a series of n independent, equally spaced measurements of range over a time T, each with rms error δ_r given by Eq. 6. Combining Eqs. 6 and 21, we have

$$\begin{aligned} \delta_{r_0} &= \frac{3c}{4\beta\sqrt{n\mathcal{R}}} \\ \delta_v &= \frac{\sqrt{3}c}{\beta T\sqrt{n\mathcal{R}}} \\ \delta_a &= \frac{6\sqrt{5}c}{\beta T^2\sqrt{n\mathcal{R}}} \end{aligned} \tag{23}$$

In these last two expressions for δ_v and δ_a, it is instructive to use (in place of β) the effective bandwidth of the signal taken about zero frequency as reference, i.e., calculate β in Eq. 3 with $f_0 = 0$. We shall call this bandwidth β_0. For narrowband signals, $\beta_0 \approx 2\pi f_0$. The range accuracy corresponding to the bandwidth β_0 corresponds to the "fine structure" information in the signal obtained from a measurement of carrier phase (reference 2). The range accuracy from this fine structure information is usually quite accurate, but is also highly ambiguous. These ambiguities do not imply that fine range information is always useless however, because it may be possible to combine sets of ambiguous range measurements to obtain non-ambiguous estimates of v and a by virtue of *a priori* information about v and a. Replacing β by $2\pi f_0$ in Eq. 23 gives

$$\begin{aligned} \delta_v &= \frac{\sqrt{3}\lambda}{2\pi T\sqrt{n\mathcal{R}}} \\ \delta_a &= \frac{3\sqrt{5}\lambda}{\pi T^2\sqrt{n\mathcal{R}}} \end{aligned} \tag{24}$$

A determination of whether ambiguities exist in the above method of estimating v and a will require special consideration for the application at hand.

Using a procedure similar to the one outlined in this section, one can derive expressions for the accuracy in estimating velocity and acceleration from a series of independent equally spaced measurements of velocity, each with rms error δ_v. The results from Eq. 19 are

$$\begin{aligned} \delta_v\,(\text{Total}) &= \frac{\delta_v}{\sqrt{n}} \\ \delta_a &= \frac{2\sqrt{3}\,\delta_v}{\sqrt{n}\,T} \end{aligned} \tag{25}$$

Consider now the important problem of estimating angle of arrival and the rate of change of the angle of arrival. Over the measurement interval of interest, we assume that the angle of arrival θ may be approximated by

$$\theta = \theta_0 + \omega t \tag{26}$$

where θ_0 and ω are unknown parameters to be determined. Proceeding as before and using Eq. 14 in Eqs. 19, we have

$$\begin{aligned} \delta_{\theta_0} &= \frac{\lambda}{\mathfrak{L}\sqrt{n\mathfrak{R}}} \\ \delta_{\omega} &= \frac{2\sqrt{3}\lambda}{\mathfrak{L}T\sqrt{n\mathfrak{R}}} \end{aligned} \tag{27}$$

$\mathfrak{L}$ can of course be specialized to various cases of interest. For the case of the interferometer considered earlier, we recall that $\mathfrak{L} = \pi N\lambda$ and

$$\delta_{\omega} = \frac{2\sqrt{3}}{\pi N T\sqrt{n\mathfrak{R}}} \tag{28}$$

Note that the quantity $n\mathfrak{R}$ corresponds to the total received energy at both apertures. Though the measurement of θ_0 with the interferometer is usually highly ambiguous, our *a priori* knowledge about the maximum value which ω can have may, in certain situations of interest, eliminate the ambiguities in the estimation of ω.

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